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LETTER TO THE EDITOR

Flory theory for directed lattice animals and directed percolation

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Abstract. The free energies of directed lattice animals in good and θ-solvents, and the free energy of directed percolation are found by the use of a simple Flory-type approximation, which accounts for the inherent anisotropy of these systems. From these free energies, we obtain the upper critical dimension below which mean-field theory breaks down. We also calculate closed-form, dimension-dependent expressions for the parallel and perpendicular correlation length exponents which characterise the asymptotic cluster shapes. These exponents are in excellent agreement with existing numerical data.

Very recently, there has been considerable progress in understanding directed percolation and directed lattice animals. In directed percolation, lattice bonds may be randomly occupied, and each bond is *directed* so that connectivity or information can 'flow' in only one direction along the bond (see, e.g., Obukhov 1980, Kinzel and Yeomans 1981, Klein and Kinzel 1981, Reynolds 1981, Redner 1982a and references therein). In this sense, a directed bond is equivalent to a diode in an electrical network. The orientation of each directed bond is fixed to lie along the positive direction of a given Cartesian axis on a hypercubic lattice (figure 1). Because a particular direction is picked out, there is a global anisotropy in cluster shapes as the percolation threshold,

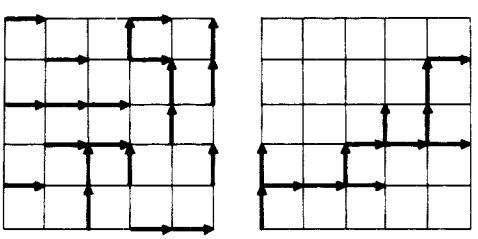

Figure 1. (*a*) Directed percolation and (*b*) directed lattice animals on the square lattice. Each directed bond connects only upwards or to the right, so that there is an anisotropy about the direction (1, 1). The longitudinal and transverse correlation lengths are defined with respect to this axis.

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p_c, is approached. As $p \to p_c^-$, the longitudinal cluster radius diverges as

$$\xi_\parallel \sim (p - p_c)^{-\bar{\nu}_\parallel} \tag{1a}$$

while the transverse cluster width diverges as

$$\xi_\perp \sim (p - p_c)^{-\bar{\nu}_\perp} \tag{1b}$$

with $\bar{\nu}_\parallel \neq \bar{\nu}_\perp$. The novel anisotropic behaviour of directed percolation has attracted considerable attention because of its relevance to several diverse topics such as Reggeon field theory (Cardy and Sugar 1980), branching Markov processes that occur in biology and irreversible chemical reactions (Grassberger and de la Torre 1979), and diffusion and conduction in systems with an external bias (Stephen 1981, Redner 1982b).

A closely related problem is that of directed lattice animals (Redner and Yang 1982, Dhar *et al* 1982). In this model, only a single cluster of directed bonds exists, and each directed bond now carries a fugacity x. As the number of bonds, N, diverges or equivalently as $x \to x_c^-$, directed animals also become extremely anisotropic in shape. Thus for the longitudinal cluster radius and the transverse cluster width, respectively, we write

$$\xi_\parallel \sim N^{\nu_\parallel} \qquad \xi_\perp \sim N^{\nu_\perp}. \tag{2}$$

One motivation for considering directed animals is the potential for connections between this model and other apparently unrelated systems. For example, for isotropic animals there exist some very intriguing and interesting connections with random field models and the Yang–Lee edge singularity (Parisi and Sourlas 1981). It is possible then that similar connections exist for directed animals. Moreover, lattice animals serve as a very simple and accurate model for dilute branched polymers in a good solvent (Lubensky and Issacson 1979; see also Stauffer 1979). Consequently, directed lattice animals may model dilute branched polymers in a suitably flowing solvent, and are also closely related to branching Markov processes with a single source. Finally, it appears that directed animals may be exactly soluble (Dhar *et al* 1982). For these reasons, the study of directed animals may prove to be quite fruitful.

In this Letter, we apply a very simple Flory-type approximation (Flory 1971, de Gennes 1979, 1980, Issacson and Lubensky 1980, Daoud and Joanny 1981) for the free energy of directed percolation and directed animals. From this, we can obtain the upper critical dimension below which mean-field theory breaks down. By minimising this free energy, we derive closed-form, dimension-dependent expressions for the parallel and perpendicular correlation length exponents. Our predictions are in excellent agreement with existing numerical data, where available.

Directed lattice animals in a good solvent. To begin, we consider the directed lattice animal or, equivalently, the directed branched polymer model. In the Flory approximation, a directed animal is assumed to have a spatially uniform density throughout; correlations are neglected (figure 2). A repulsive interaction energy arises due to the excluded-volume effect between different monomers comprising the polymer. To calculate this energy, we consider a good solvent where two-body repulsive forces dominate. Under these conditions, the potential energy is the integral of the square of the monomer density over the spatial extent of the polymer. Due to the anisotropic shape of the polymer, its volume scales as $\xi_\parallel \xi_\perp^{d-1}$. This leads to

$$U_{\text{tot}} \sim N^2 / \xi_\parallel \xi_\perp^{d-1}. \tag{3a}$$

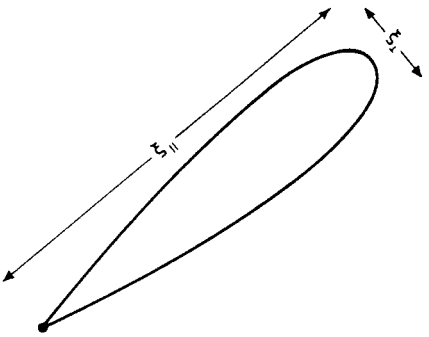

Figure 2. Schematic picture of a directed animal or a directed percolation cluster in the Flory approximation.

There are two contributions to the configurational entropy. First, there is the entropy associated with the degrees of freedom for the monomer positions in the perpendicular direction. Since $\nu_\perp = \frac{1}{4}$ in mean-field theory, there is a contribution to the entropy which is of the same form as that found for isotropic animals (see e.g. Issacson and Lubensky 1980, Daoud and Joanny 1981). The entropy term just described therefore has the form,

$$\xi_\perp^2/N^{1/2}. \tag{3b}$$

Furthermore, a second term occurs because of additional entropy associated with degrees of freedom in the longitudinal direction. In the mean-field approximation $\nu_\parallel = \frac{1}{2}$ (Redner and Yang 1982); therefore we assume a Gaussian distribution for these degrees of freedom. This gives a contribution to the entropy of the form

$$\xi_\parallel^2/N. \tag{3c}$$

Thus the free energy of the system is

$$F = U_{\text{tot}} - TS$$
$$\sim N^2/\xi_\parallel \xi_\perp^{d-1} + \xi_\parallel^2/N + \xi_\perp^2/N^{1/2}. \tag{4a}$$

Minimising this expression with respect to both $\xi_\parallel$ and $\xi_\perp$ yields two simple conditions involving both the exponents $\nu_\parallel$ and $\nu_\perp$. Solving, we find

$$\nu_\parallel = (d+11)/4(d+2) \qquad \nu_\perp = 9/4(d+2). \tag{4b}$$

Notice that these expressions give the correct mean-field limits $\nu_\parallel = \frac{1}{2}$ and $\nu_\perp = \frac{1}{4}$ in $d = 7$ (Redner and Yang 1982). For $d > 7$, the repulsive energy term vanishes as $N \to \infty$, and the mean-field exponents are correct. However, below seven dimensions the excluded-volume repulsion leads to a breakdown of the mean-field theory.

The numerical values of the exponents, together with a comparison with recent series data of Redner and Yang (1982), is given in table 1. The agreement is quite good in low dimensions, but the agreement becomes progressively worse in higher dimensions. However, the relative shortness of the series in higher dimensions may lead to inaccurate extrapolations for the exponents. It is particularly striking that the numerical and Flory estimates coincide at $d = 3$. We suspect that this may be an exact result, since the Flory theory for the isotropic animal problem gives the exact value

Table 1. Comparison of the values for $\nu_\parallel$ and $\nu_\perp$ for directed lattice animals from the Flory theory (equation (4*b*)) and from the series data

	$\nu_\parallel$		$\nu_\perp$	
d	Flory	Series	Flory	Series
2	0.8125	0.800±0.001	0.5625	0.500±0.003
3	0.70	0.700±0.001	0.45	0.450±0.005
4	0.625	0.64±0.01	0.375	0.40±0.02
5	0.5714...	0.62±0.02	0.3214	0.39±0.04
6	0.53125	0.61±0.03	0.28125	0.35±0.05
7	0.50	0.60±0.05	0.25	0.35±0.08

in $d=3$ (Parisi and Sourlas 1981). In light of their recent work in which the isotropic lattice animal problem in $d=3$ is mapped to the exactly soluble Yang–Lee edge singularity problem in $d=1$, it may be worthwhile to explore whether similar connections exist for directed animals.

Directed animals in a θ-solvent. The approach given above can be extended to directed branched polymers in a θ-solvent as well. In such a solvent, two-body forces vanish and the repulsion is governed by a three-body term. The repulsive energy is therefore the integral of the cube of the concentration over the spatial extent of the polymer, while the entropy is the same as in equation (4*a*). This leads to a free energy of the form

$$F = N^3/\xi_\parallel^2\xi_\perp^{2d-2} + \xi_\parallel^2/N + \xi_\perp^2/N^{1/2}. \tag{5a}$$

A minimisation of this free energy with respect to $\xi_\parallel$ and $\xi_\perp$ yields

$$\nu_\parallel = (d+7)/4(d+1) \qquad \nu_\perp = 3/2(d+1). \tag{5b}$$

Here we find that mean-field theory extends down to five dimensions, below which corrections in the exponents should appear. This treatment can be easily extended to consider the general case of a solvent where the repulsion is governed by an n-body force.

Directed percolation. To treat directed percolation, we note that the presence of many clusters in the system leads to an inward 'pressure' on a given cluster. This effect is most easily calculated in terms of a screening of the monomer–monomer repulsion (Edwards 1966, de Gennes 1980). The screening effect is found to reduce the repulsion by the inverse of the number average cluster size, S, which scales as $N^{\gamma/(\beta+\gamma)}$. Since for directed percolation the mean-field exponents are $\beta=\gamma=1$, the screening reduces the repulsive energy term by a factor $N^{-1/2}$. This same result also occurs in isotropic percolation since the 'magnetic' exponents β and γ are the same for both directed and isotropic percolation in mean-field theory. These considerations lead us to the following free energy:

$$F = N^{3/2}/\xi_\parallel\xi_\perp^{d-1} + \xi_\parallel^2/N + \xi_\perp^2/N^{1/2}. \tag{6a}$$

Minimising this expression gives

$$\nu_\parallel = (d+9)/4(d+2) \qquad \nu_\perp = 7/4(d+2). \tag{6b}$$

These formulae give $\nu_\parallel = \frac{1}{2}$ and $\nu_\perp = \frac{1}{4}$ for $d=5$, the upper critical dimension of the

system (Obukhov 1980, Redner 1982a). Below this dimension corrections to mean-field theory will appear.

We emphasise that the exponents obtained characterise the divergence of the cluster radii on N. To convert to the exponents $\bar{\nu}_\parallel$ and $\bar{\nu}_\perp$ describing the divergence as a function of $p - p_c$, we must multiply the expressions of equation (6*b*) by a factor of $\beta\delta = \beta + \gamma$. This can be done in two dimensions where numerical estimates for the requisite exponents are available. In table 2 we summarise the results of the Flory theory together with the current numerical estimates in $d = 2$. The agreement is excellent in two dimensions, and it would be very interesting to obtain numerical data for the correlation length exponents in three and four dimensions in order to test the Flory theory further.

Table 2. Comparison of the values for $\nu_\parallel$ and $\nu_\perp$ for directed percolation from the Flory theory (equation (6*b*)) and from recent numerical work.

	$\nu_\parallel$		$\nu_\perp$	
d	Flory	Numerical	Flory	Numerical
2	0.6875	0.678 ± 0.004†	0.4375	0.435 ± 0.004†
3	0.60	—	0.35	—
4	0.5416...	—	0.2916...	—
5	0.50	—	0.25	—

† We use the values of Blease (1977a, b, c), $\beta = 0.28$, $\gamma = 2.27$ in two dimensions and the values $\bar{\nu}_\parallel = 1.73 \pm 0.01$ and $\bar{\nu}_\perp = 1.11 \pm 0.01$ (see, e.g., Kinzel and Yeomans 1981, Dhar and Barma 1981, Essam and De'Bell 1981) to obtain the 'numerical' estimates for the exponents $\nu_\parallel = \bar{\nu}_\parallel/(\beta + \gamma)$ and $\nu_\perp = \bar{\nu}_\perp/(\beta + \gamma)$.

In summary, the Flory theory works remarkably well in predicting the asymptotic size and shape of directed lattice animals and directed percolation clusters. In spite of the mean-field character of the theory and the uncontrolled approximations it entails, it is possible to obtain quite accurate results for a wide range of problems with minimal effort. Thus the Flory theory appears to possess a generality far beyond the scope of its original applications. Consequently, it appears worthwhile to attempt to apply the Flory theory to as wide a range of problems as possible. By these means, we may find the limits of validity for this theory and perhaps gain a better insight into its underlying basis.

After this work was completed, we received a preprint by Day and Lubensky who studied directed animals in a good solvent (excluding loop formation) using field theory. They find an upper critical dimension of seven, and calculate exponents to lowest order in an ε-expansion ($\varepsilon = 7 - d$). It is interesting to compare their results, and the field-theory results of Obukhov (1980) for directed percolation, with those of the Flory theory to first order in ε.

		Flory theory	Field theory
Directed animals: (good solvent)	$\nu_\parallel$	$1/2 + \varepsilon/36$	$1/2 + \varepsilon/24$
	$\nu_\perp$	$1/4 + \varepsilon/36$	$1/4 + \varepsilon/36$
Directed percolation:	$\nu_\parallel$	$1/2 + \varepsilon/28$	$1/2 + \varepsilon/12$
	$\nu_\perp$	$1/4 + \varepsilon/28$	—

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Note added in proof. After our work was accepted for publication we received a preprint from Lubensky and Vannimenus in which they derive results essentially the same as ours.

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